Synopsis

on

SOLUTION OF INTEGRAL AND DIFFERENTIAL EQUATIONS USING DIFFERENTIAL TRANSFORM METHOD AND ITS APPLICATION

by

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Title of Thesis: "Solution of integral and differential equations using differential transform method and its application"

A remarkable, effective, reliable semi-analytical numerical metho is Differential Transform Method (DTM). Various types of differential and integral equations solved by this remarkable method. By this method, we easily get the solution of the equations with accuracy. Discretization, linearization or small perturbations are not essential for this method. As a result, the numerical computation becomes remarkable simple. J. K. Zhou is the pioneer of the concept of DTM. At first, this concept was successfully applied for solve linear and nonlinear IVP's in electric circuit analysis, in 1986 and get the remarkable result. DTM based on the Tailors series expansion that provides solution in the polynomial form.

Ordinary and partial differential equations are solved by various numerical and analytical methods. Laplace or Fourier transformation transforms the ordinary differential equations into a corresponding algebraic equation, which are actually integral transform method. To find series solutions for complex differential equations often becomes very complex and time consuming. Several techniques are available in the literature, namely Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM), Adomin Decomposition Method (ADM), Variational Iteration Method (VIM), etc. All of these methods ensure the semi- analytical, namely, series solution of complex differential equations. Solving initial value problems can be quite straightforward when using these methods. Dealing with boundary conditions becomes more challenging when we have to include boundary values in our initial estimates. When the boundary conditions are highly complex, incorporating them becomes challenging and can often introduce errors into the solution, making it difficult to manage. Numerical methods are incredibly useful for solving linear differential equations. However, their efficacy diminishes as the level of non-linearity increases. In the event of non-linearity, the problem must be solved using the implicit method, as the explicit method is not effective in such cases. But often it incorporates errors in the calculation. Removing the error is quite difficult often.

In 2008, Odibat presented essential definitions and basic operations of the one-dimensional differential transform method, along with its applications to a range of

differential and integral equations.DTM applied to solve different types of mathematical problems. Khatib, A. (2016) gave the idea about utility of DTM for Differential Equations.Chen and Ho (1996) applied differential transform method in solving the eigenvalue problems. They applied DTM to solve Sturm-Liouville problems and get eigenvalues and eigenfunctions. They solved two eigen value problems and compared the results obtained by another analytical method. Hasan (2002) used DTM to solve the eigenvalue problems and get the remarkable results. He used Differential Transform Method on the one-dimensional planar Bratu problems, known as nonlinear eigenvalue problem in 2007.Jang and Chen in 2000 applied DTM to simplify the linear and non- linear initial value problems. Hassan in 2004 also solved some higher order initial value problems. After that in 2008, he presented a comparisonbetween two technics of solution DTM and ADM. Bachir Nour Kharrat and George Toma in 2019 given general framework of differential transform method for solving strongly non-linear initial value problems represented by ODE. Sarita Devi and Manjeet Jakhar in 2022 used DTM in solving the initial value problems.

Chen and Ho (1999) solved the partial differential equation with the help of two-dimensional differential transform method to get the series solution. Ayaz (2003) solved the initial value problems, which is applicable for PDE and showed that this result is better than the result of decomposition method. Alquran et al. (2010) solve non-homogeneous linear PDE with the help of Laplace DTM. Kumari (2015) solve wave equations with the help of Laplace DTM.Tari et al. (2009) solved Volterra integral equations. Non-linear Volterra integro-differential equations are solved by Kajani and Shehni (2011). Darania et. al.(2011) solved two-dimensional non-linear Volterra integro-differential equations.Ayaz (2004) and Hassan (2008) solved system of differential equations with the help of Differential Transform Method.

Initially I started solving the fractional-order integral equation using the differential transform method. In this study, we employed a generalized differential transform method to effectively address various types of fractional-order integral equations. All the fractional integrals are written in the Riemann–Liouville sense, and fractional derivatives are written in the Caputo sense.

Upon completing the analysis of integral equations, I aim to investigate the feasibility of solving certain specific differential equations that could have

practical applications in real life.Lane-Emden-type differential equations are considered and solved by using the differential transform method. The temperature in a self-gravitating star has been widely described in the Lane-Emden differential equation. The Lane-Emden differential equation serves as a quintessential model for evaluating innovative mathematical and numerical methods aimed at resolving nonlinear equations. This study effectively addresses the challenges of singular initial value problems of the Lane-Emden type using the differential transform method. This method is applicable to solving various linear and nonlinear problems, which decreases computational work. Here, I introduce numerical problems to describe the high accuracy and efficiency of our proposed method.

Now, following the same path, the author wants to solve another named problem, i.e., linear and nonlinear Goursat problems. This article presents the Differential Transform Method (DTM) as a promising and highly effective approach for addressing a range of complex problems, praised for its simplicity and ease of calculation. We applied this constructive method to solve linear and nonlinear hyperbolic partial differential equations, namely the Goursat problem. To validate the accuracy and applicability of the method, some useful examples have been solved. The accuracy level of the obtained results exposes the acceptability of this method over existing numerical methods. Solving the named equations offers fresh perspectives and applications within the already established field. The equations in question can typically be solved using a variety of mathematical techniques. So, solving those equations using the semi-analytical differential transform method can give a new insight into previously established studies.

In my next work, I solved a Riccati differential equation using the differential transform method. The Differential Transform Method (DTM) is a highly effective numerical technique recognized for its ability to solve the Riccati differential equation, a nonlinear first-order equation with numerous applications in engineering and science. To validate the proposed method concerning the equation, we explore three illustrative examples that include both constant and variable coefficients. Additionally, to assess the method's accuracy, the results were visually compared with the exact solutions.

After conducting an in-depth analysis of the differential transform method and confirming the results through comparison with existing literature, I set out to

implement this technique within a contaminant flow model. I intend to conduct a comprehensive mathematical analysis of groundwater contamination within physical systems by employing the differential transform method. This research presents the development of a mathematical modeling framework designed to evaluate the impacts of groundwater contamination in aquifer-aquitard systems across diverse geological formations.Different mathematical functions are used to depict dispersion and velocity in different media. Boundary conditions are often unpredictable due to complex geometry and the nature of the aquifer. Initial conditions are quite easy to obtain, so different types of initial conditions have been considered. Advection-dispersion equation (ADE) has been used to model the contaminant concentration in the aquifer-aquitard system. The Differential Transform Method (DTM) not only significantly reduces computational effort but is also an effective tool for solving both linear and nonlinear initial-value partial differential equations (PDEs). Two-dimensional DTM has been used to solve the advection-dispersion equation semi-analytically, and MATLAB was used to plot the solution. Variations in dispersion, velocity, and initial conditions have distinct effects on the contaminant concentration profile, while the influences of space and time are also prominently evident.

In most problem solved using the differential transform method, initial conditions are needed to solve the problem. So, our target is to use the boundary conditions while solving real-life problems. So, in the next problem, I solved the dispersion equation along with boundary values using the differential transform method. Which is not only a new concept for ADE but also adds some value in describing boundary value problems using DTM. Groundwater contamination is a major threat to society, mostly when it occurs from non-point sources. The easiest way to describe a non-point source is to model the phenomenon mathematically. In this study, an effective computational method is developed to solve the boundary value problem. Identifying the initial condition for a problem may appear straightforward initially; however, the true difficulty often lies in establishing the correct boundary conditions and tackling the related boundary-value problems. The movement of fluids through a porous medium has been effectively represented using the advection-dispersion equation (ADE), taking into account the relevant boundary conditions. To solve this complex problem, a powerful technique known as the

Differential Transform Method (DTM) has been employed. The results of this study show that the differential transform method can achieve a unique result.

Finally, I conclude that the DTM is very useful for solving nonlinear differential equations, as well as being implemented in solve real-world problems. The results achieved closely align with previously established findings regarding named equations and have practical applications for tackling real-world challenges, first two chapters mainly focused on the literature review and discussion of mathematical methods.The chapter wise conclusions are discussed as follows:

From chapter-3 I can able to conclude that,I employed a generalized differential transform method to derive numerical solutions for the singular integral equation of fractional order. I have provided several examples demonstrating the effectiveness of the proposed method. The proposed mechanism demonstrates excellent performance, ease of application, and impressive accuracy.

Chapter-4 can conclude that Lane-Emden type equations have been successfully solved using the Differential Transform Method (DTM), demonstrating its effectiveness as a reliable tool. Furthermore, the distinct initial value problem associated with Lane-Emden equations was successfully tackled through the application of DTM. The DTM streamlines complex calculations and simplifies the process through straightforward manipulations. Some problems were solved by using this method and it gave remarkable results.

In concluding Chapter-5, I introduce an analytical framework for solving linear and non-linear Goursat problems, yielding a unique result that is easier to calculate than traditional numerical methods. This innovative and visually appealing approach offers a simpler solution for mathematical calculations, while also presenting significant opportunities for leveraging the Two-Dimensional Differential Transform method in future research endeavors. All calculations can be performed through a simple procedure. The Two-dimensional Differential Transform Method (DTM) was tested with several examples, demonstrating its simplicity and accuracy. This review seeks to explore the recent advancements in mathematics and highlights the crucial role that DTM plays in addressing a variety of complex challenges. I am confident that this innovative DTM method surpasses existing techniques in both reliability and accuracy.

Concluding chapter-6, it is worth showing that I employed the Differential Transform Method to effectively solve the Riccati Differential Equation, addressing both variable and constant coefficients. DTM is a highly effective and dependable method. The efficiency of DTM enhances its applicability, indicating a need for further implementation and exploration. The scale of computations and the speed at which they converge to exact solutions demonstrate the effectiveness of this method in addressing a range of physical problems. I assess the accuracy of this method by comparing the results with the exact solutions visually.

Next chapter DTM method has been employed to solve contaminant transport through porous medium and from the study I can able to conclude that: Depending upon the dispersion and velocity the contaminant concentration profile changes and when the dispersion and velocity depends on both space and time the impact is maximum. In case of various geometries, the impact is different for different zones. Differential transformation method is easy to understand and quite helpful for solving this kind of partial differential equation-based problems and physical systems. Differential transform method is unconditionally stable for small values of time and space so this solution may be quite helpful to benchmark the system.

Applying DTM for solving boundary value problem and applying it in contaminant transport through porous medium I can able to conclude that: Contaminant concentration increases with increasing time and may show its adverse impact on human health. The level of contaminant concentration increases with increasing distance and after a certain distance the level of contamination started decreasing. DTM can handle the boundary value problem and provides more or less good results. Present problems can later serve as benchmarks for semi-analytical solutions.

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